

# Hyperquasivarieties

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**ABSTRACT.** We consider the notion of hyper-quasi-identities and hyperquasivarieties, as a common generalization of the concept of quasi-identity and quasivariety invented by A. I. Mal'cev, cf. [10], cf. [5] and hypervariety invented by the authors in [6].

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## 1. Notations

An identity is a pair of terms where the variables are bound by universal quantifiers. Let us take the following medial identity as an example

$$\forall u \forall x \forall y \forall w (u \cdot x) \cdot (y \cdot w) = (u \cdot y) \cdot (x \cdot w).$$

Let us look at the following hyperidentity

$$\forall F \forall u \forall x \forall y \forall w F(F(u, v), F(x, y)) = F(F(u, x), F(v, y)).$$

The hypervariable  $F$  is considered in a very specific way. Firstly every hypervariable is restricted to functions of a given arity. Secondly  $F$  is restricted to term functions of the given type. Let us take the variety  $A_{n,0}$  of abelian groups of finite exponent  $n$ . Every binary term  $t \equiv t(x, y)$  can be presented by  $t(x, y) = ax + by$  with  $a, b \in \mathbb{N}_0$ . If we substitute the binary hypervariable  $F$  in the above hyperidentity by  $ax + by$ , leaving its variables unchanged, we get

$$a(au + bv) + b(ax + by) = a(au + bx) + b(av + by).$$

This identity holds for every term  $t(x, y) = ax + by$  for the variety  $A_{n,0}$ . Therefore we say that the hyperidentity holds for the variety  $A_{n,0}$ .

## 2. Hyper-quasi-identities

In the sequel we use the definition of *hyperterm* from [6]. We accept the notation of [2], [8], [5] and [9].

We recall only our definitions of [6] of the fact that a hyperidentity is satisfied in an algebra of a given type and the notion of hypervariety:

**Definition 2.1.** An algebra  $\mathbf{A}$  satisfies a hyperidentity  $h_1 = h_2$  if for every substitution of the hypervariables by terms (of the same arity) of  $\mathbf{A}$  leaving the variables unchanged, the identities which arise hold in  $\mathbf{A}$ . In this case, we write

$\mathbf{A} \models (h_1 = h_2)$ . A variety  $V$  satisfies a hyperidentity  $h_1 = h_2$  if every algebra in the variety does; in this case, we write  $V \models (h_1 = h_2)$ .

**Definition 2.2.** A class  $V$  of a algebras of a given type is called a hypervariety if and only if  $V$  is defined by a set of hyperidentities.

The following was proved in [6]:

**Theorem 2.3.** A variety  $V$  of type  $\tau$  is defined by a set of hyperidentities if and only if  $V = HSPD(V)$ , i.e.  $V$  is a variety closed under derived algebras of type  $\tau$ .

We recall from [10] and [5]:

**Definition 2.4.** A quasi-identity  $e$  is an implication of the form:

$$(t_0 = s_0) \wedge \dots \wedge (t_{n-1} = s_{n-1}) \rightarrow (t_n = s_n).$$

where  $t_i = s_i$  are  $k$ -ary identities of a given type, for  $i = 0, \dots, n$ .

A quasi-identity above is *satisfied in an algebra*  $\mathbf{A}$  of a given type if and only if the following implication is satisfied in  $\mathbf{A}$ : given a sequence  $a_1, \dots, a_k$  of elements of  $A$ . If this elements satisfy the equations  $t_i(a_1, \dots, a_k) = s_i(a_1, \dots, a_k)$  in  $\mathbf{A}$ , for  $i = 0, 1, \dots, n-1$ , then the equality  $t_n(a_1, \dots, a_k) = s_n(a_1, \dots, a_k)$  is satisfied in  $\mathbf{A}$ . In that case we write:

$$\mathbf{A} \models (t_0 = s_0) \wedge \dots \wedge (t_{n-1} = s_{n-1}) \rightarrow (t_n = s_n).$$

A quasi-identity  $e$  is *satisfied in a class*  $V$  of algebras of a given type, if and only if it is satisfied in all algebras  $\mathbf{A}$  belonging to  $V$ .

We modify the definition above in the following way:

**Definition 2.5.** A hyper-quasi-identity  $e^*$  is an implication of the form:

$$(T_0 = S_0) \wedge \dots \wedge (T_{n-1} = S_{n-1}) \rightarrow (T_n = S_n).$$

where  $T_i = S_i$  are hyperidentities of a given type, for  $i = 0, \dots, n$ .

A hyper-quasi-identity  $e^*$  is hyper-satisfied (holds) in an algebra  $\mathbf{A}$  is and only if the following implication is satisfied:

If  $\sigma$  is a hypersubstitution of type  $\tau$  and the elements  $a_1, \dots, a_n \in A$  satisfy the equalities  $\sigma(T_i)(a_1, \dots, a_k) = \sigma(S_i)(a_1, \dots, a_k)$  in  $\mathbf{A}$ , for  $n = 0, 1, \dots, n-1$ , then the equality  $\sigma(T_n)(a_1, \dots, a - k) = \sigma(S_n)(a_1, \dots, a_k)$  holds in  $\mathbf{A}$ .

By other words, hyper-quasi-identity is a universally closed Horn  $\forall x \forall \sigma$ -formulas, where  $x$  vary over all sequences of individual variables (occurring in terms of the implication) and  $\sigma$  vary over all hypersubstitutions of a given type.

**Remark.** All hyper-quasi-identities and hyperidentities are written without quantifiers but they are considered as universally closed Horn  $\forall$ -formulas (cf. [10]). A syntactic side of the notions described here will be considered in a forthcoming paper.

### 3. Examples of hyper-quasi-identities

#### 3.1. Hyper-quasi-identities for abelian algebras.

**Definition 3.1.** An algebra  $\mathbf{A}$  is called abelian if for every  $n > 1$  and every  $n$ -ary term operation  $f$  of  $\mathbf{A}$  and for all  $u, v, x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1}$  the following equivalence holds:

$$f(u, x_1, \dots, x_n) = f(u, y_1, \dots, y_n) \leftrightarrow f(v, x_1, \dots, x_n) = f(v, y_1, \dots, y_n)$$

A variety  $V$  is called abelian, if each algebra of  $V$  is abelian.

It follows from [9, p. 40], [4, p. 290]:

**Proposition 3.2.** *An algebra  $\mathbf{A}$  is abelian if and only if the following hyper-quasi-identity holds in  $\mathbf{A}$ :*

$$F(u, x_1, \dots, x_n) = F(u, y_1, \dots, y_n) \leftrightarrow F(v, x_1, \dots, x_n) = F(v, y_1, \dots, y_n).$$

**Example.** The variety  $RB$  of rectangular bands fulfills the hyperidentities of type  $(1, 2, 3, \dots, n, \dots)$ :

$$F(x, \dots, x) = x, F(F(x_{11}, \dots, x_{1n}), x_2, \dots, x_n) = F(x_{11}, x_2, \dots, x_n), \\ F(F(x_1, \dots, x_{n-1}), F(x_{n1}, \dots, x_{nn})) = F(x_1, \dots, x_{n-1}, x_{nn}).$$

We can derive by the associative hyperidentity:  $F(x, F(y, z)) = (F(x, y), z)$ , that the following hyper-quasi-identity holds in  $RB$ :

$$F(u, x_1, \dots, x_n) = F(u, y_1, \dots, y_n) \rightarrow F(v, u, x_1, \dots, x_{n-1}) = F(v, u, y_1, \dots, y_{n-1}) \rightarrow \\ F(v, x_1, \dots, x_n) = F(v, y_1, \dots, y_n), \text{ i.e. the variety } RB \text{ is abelian.}$$

**Remark.** For further examples of abelian varieties consult [9] and [13].

**3.2. Semidistributive lattices.** A lattice is *joinsemidistributive* if it satisfies the following condition, cf. [9, p. 82], [5, p. 141]:

$$(SD_{\vee}) \ x \vee y = x \vee z \text{ implies } x \vee y = x \vee (y \wedge z)$$

The *meetsemidistributivity* is defined by duality:

$$(SD_{\wedge}) \ x \wedge y = x \wedge z \text{ implies } x \wedge y = x \wedge (y \vee z)$$

A lattice is *semidistributive* if it is simultaneously join and meet semidistributive.

**Proposition 3.3.** *Let  $\mathbf{L} = (L, \wedge, \vee)$  be a lattice which is semidistributive. Then the following hyper quasi-identity is hypersatisfied in  $\mathbf{L}$ :*

$$(F(x, y) = F(x, z)) \rightarrow (F(x, y) = F(x, G(y, z)))$$

*Proof.* We consider all cases on hypervariables of a semidistributive lattice.

Case 1.  $F(x, y) := x$ .

Obviously  $(x = x) \rightarrow (x = x)$ .

Case 2.  $F(x, y) := y$ . Consider  $G(y, z) := y, z, y \wedge z, y \vee z$ .

Then the following quasi-identities are satisfied in  $\mathbf{L}$ :

$$(y = z) \rightarrow (y = y), (y = z) \rightarrow (y = z), (y = z) \rightarrow (y = y \wedge z), \\ (y = z) \rightarrow (y = y \vee z).$$

Case 3.  $F(x, y) := x \wedge y$ . Consider  $G(y, z) := y, z, y \wedge z, y \vee z$ .

Then the following quasi-identities hold in  $\mathbf{L}$ :

$$(x \wedge y = x \wedge z) \rightarrow (x \wedge y = x \wedge y), (x \wedge y = x \wedge z) \rightarrow (x \wedge y = x \wedge z), \\ (x \wedge y = x \wedge z) \rightarrow (x \wedge y = x \wedge (y \wedge z)), (x \wedge y = x \wedge z) \rightarrow (x \wedge y = x \wedge (y \vee z)).$$

The last one follows from the meet semidistributivity of  $\mathbf{L}$ .

Case 4.  $F(x, y) := x \vee y$ . Consider  $G(y, z) := y, z, y \wedge z, y \vee z$ .

Then the following quasi-identities hold in  $\mathbf{L}$ :

$(x \vee y = x \vee z) \rightarrow (x \vee y = x \vee y), (x \vee y = x \vee z) \rightarrow (x \vee y = x \vee z),$   
 $(x \vee y = x \vee z) \rightarrow (x \vee y = x \vee (y \wedge z)), (x \vee y = x \vee z) \rightarrow (x \vee y = x \vee (y \vee z)).$  The prelast one follows from the join semidistributivity of  $\mathbf{L}$ .  $\square$

#### 4. Derived algebras

We recall from [6], cf. [2, p. 145] the notion of a *derived algebra* and the *derived class* of algebras. Given a type  $\tau = (n_0, n_1, \dots, n_\gamma, \dots)$ . An algebra  $\mathbf{B}$  of type  $\tau$  is called a derived algebra of  $\mathbf{A} = (A, f_0, f_1, \dots, f_\gamma, \dots)$  if there exist term operations  $t_0, t_1, \dots, t_\gamma, \dots$  of type  $\tau$  such that  $\mathbf{B} = (A, t_0, t_1, \dots, t_\gamma, \dots)$ . For a class  $K$  of algebras of type  $\tau$  we denote by  $\mathbf{D}(K)$  the class of all derived algebras (of type  $\tau$ ) of  $K$ . A class  $K$  is called *deriverably closed* if and only if  $\mathbf{D}(K) \subseteq K$ .

For a given algebra  $\mathbf{A}$  we denote by  $QId(\mathbf{A})$  and  $HQId(\mathbf{A})$  the set of all quasi-identities and hyper-quasi-identities satisfied (hypersatisfied) in  $\mathbf{A}$ , respectively. Similarly for a class  $K$ ,  $QId(K)$  and  $HQId(K)$  denote the set of all quasi-identities and hyper-quasi-identities satisfied (hypersatisfied) in  $K$ , respectively. Following [5] by  $\mathbf{Q}(K)$  and  $H\mathbf{Q}(K)$  we denote the class of all algebras of a given type satisfying (hypersatisfying) all the quasi-identities and hyper-quasi-identities of  $K$ , respectively. The transformation  $T$  of a quasi-identity  $e$  into hyper-quasi-identity  $e^*$  is defined in a natural way. Similarly  $T^{-1}$  transform every hyper-quasi-identity  $e^*$  into the quasi-identity  $e$ . (cf. [6]).

**Proposition 4.1.** *Given a class  $K$  of algebras of type  $\tau$ . Then the following equality holds:*

$$HQId(K) = T(QId(\mathbf{D}(K))).$$

*Proof.* To prove the inclusion  $\subseteq$ , given a hyper-quasi-identity  $e^*$  of  $K$  and an algebra a derived algebra  $\mathbf{B} = \mathbf{A}^\sigma$  for  $\mathbf{A} \in K$ . Then by definition 2.5  $T^{-1}(e)$  is satisfied in  $\mathbf{B}$  as a quasi-identity, i.e.  $e^* \in T(QId(\mathbf{D}(K)))$ .

For a proof of the converse inclusion, given a quasi-identity  $e$  satisfied in  $\mathbf{D}(K)$  and an algebra  $\mathbf{A}$  of  $K$ . Let  $a_1, \dots, a_k \in A$ . Given an hypersubstitution  $\sigma$  of type  $\tau$  and consider  $\sigma(e)$  in  $\mathbf{A}$ . As  $\mathbf{A} \in \mathbf{D}(K)$ , then  $\sigma(e)$  may be considered as a quasi-identity of the derived algebra  $\mathbf{A}^\sigma$  (cf. [12]). Assume that  $p_i^{\mathbf{A}^\sigma}(a_1, \dots, a_k) = q_i^{\mathbf{A}^\sigma}(a_1, \dots, a_k)$ , therefore as  $\mathbf{A} \in \mathbf{D}(K)$  we obtain that  $p_n^{\mathbf{A}^\sigma}(a_1, \dots, a_k) = q_n^{\mathbf{A}^\sigma}(a_1, \dots, a_n)$ , i.e. that the  $T(e)$  holds in  $K$  as a hyper-quasi-identity, i.e.  $T(e) \in HQId(K)$ .  $\square$

The role of derived algebras in solvability questions may be visualized by the following:

**Theorem 4.2.** *Given a locally finite variety  $V$ . The class  $K$  of all locally solvable algebras in  $V$  is a hypervariety.*

*Proof.* By corollary 7.6, of [9] the class of all locally solvable algebras in a locally finite variety  $V$  is a variety. Due to our theorem 2.3 we need to show that  $K$  is deriverably closed. Given a derived algebra  $\mathbf{B}$  of an algebra  $\mathbf{A} \in K$  and its finite

subalgebra  $\mathbf{F}^*$ . Then  $\mathbf{F}^*$  is a derived subalgebra of a finite subalgebra  $\mathbf{F}$  of  $\mathbf{A}$ , i.e.  $\mathbf{F}^* = \mathbf{F}^\sigma$ , for a hypersubstitution  $\sigma$ . From theorem 5.7 of [12] we conclude, that as  $\mathbf{F}$  is solvable, therefore  $\mathbf{F}^\sigma$  is solvable.  $\square$

**Remark.** The assumption that  $V$  is locally finite is essential, as for the variety of abelian groups  $AG$ , which are locally solvable, it is easy to notice that  $AG$  does not constitute a hypervariety, as the abelian law:  $xy = yx$  is not satisfied in  $AG$  as a hyperidentity.

**4.1. Quasicompact classes.** We accept the definition of a *quasicompact classes* invented by V. A. Gorbunov [5, p. 77] and his theorem 2.3.1, which states:

**Theorem 4.3.** *For a class  $K$ , the prevariety  $\mathbf{SP}(K)$  is a quasivariety if and only if  $K$  is quasicompact.*

We reformulate the definition of *prevariety* to a *hyperprevariety* and theorem above for the case of hyper-quasi-identities:

**Definition 4.4.** For a class  $K$  of algebras, the class  $\mathbf{SP}(\mathbf{D}(K))$  (for short  $\mathbf{SPD}(K)$ ) will be called hyperprevariety generated by  $K$ .

A class  $K$  of algebras of a given type is hyper-quasi-compact provided that from the infinite implication, indexed by  $I$  and hyper-satisfied in  $K$  it follows that for some finite subset  $F \subseteq I$  the finite implication (restricted to  $F$ ) is hypersatisfied in  $K$ .

**Theorem 4.5.** *For a class  $K$ , the prevariety  $\mathbf{SP}(\mathbf{D}(K))$  is a quasivariety if and only if  $\mathbf{D}(K)$  is quasicompact if and only if  $K$  is hyper-quasi-compact.*

*Proof.* The first equivalence is the V. A. Gorbunov theorem of [5, p. 77].

The second equivalence follows from the proposition 4.1.  $\square$

## 5. Hyper-quasi-varieties

We reformulate the notion of *quasivariety* invented by A. I. Mal'cev in [10, p. 210] for the case of *hyper-quasi-identities* of a given type in a natural way:

**Definition 5.1.** A class  $K$  of algebras of type  $\tau$  is called a hyperquasivariety if there is a set  $\Sigma$  of hyper-quasi-identities of type  $\tau$  such that  $K$  consists exactly of those algebras of type  $\tau$  that hypersatisfy all the hyper-quasi-identities of  $\Sigma$ .

From proposition 4.1 we obtain immediately:

**Theorem 5.2.** *A quasivariety  $K$  of algebras given type is a hyperquasivariety if and only if it is derivably closed.*

In the sequel we use the standard notation:  $\mathbf{S}$  for the operator of creating *subalgebras*,  $\mathbf{P}$ ,  $\mathbf{P}_s$ ,  $\mathbf{P}_r$ ,  $\mathbf{P}_u$  and  $\mathbf{P}\omega$  for *products*, *subdirect products*, *reduced products*, *ultraproducts* and *direct products of finite families of structures*, respectively.  $\mathbf{L}$  and  $\mathbf{L}_s$  will denote the operators of *direct limits* and *superdirect limits*.

Following A. I. Mal'cev [10, p. 153, 215] for a given class  $K$  of algebras of a given type, by  $\mathbf{S}(K)$ ,  $\mathbf{P}(K)$ ,  $\mathbf{P}_s(K)$ ,  $\mathbf{P}_r(K)$ ,  $\mathbf{P}_u(K)$  we denote the class of algebras isomorphic to all possible subalgebras, direct products, subdirect products, reduced (filtered) products or ultraproducts of algebras of  $K$ , respectively.  $\mathbf{P}_\omega$  is the class of algebras isomorphic to the direct products of finite families of structures of  $K$ . Similarly,  $\mathbf{L}(K)$  and  $\mathbf{L}_s(K)$  will be the class of algebras isomorphic to direct and superdirectlimits of algebras of  $K$ , respectively (cf. [5, p.21]).

Adding a trivial system to  $K$  we obtain the class  $K_0$ .

We reformulate the resut of [6]:

**Proposition 5.3.** *Given a class  $K$  of algebras, then the following inclusions holds:*

- 1)  $\mathbf{DS}(K) \subseteq \mathbf{SD}(K)$ ;
- 2)  $\mathbf{DP}(K) \subseteq \mathbf{PD}(K)$ ;
- 3)  $\mathbf{DP}_\omega(K) \subseteq \mathbf{P}_\omega\mathbf{D}(K)$ ;
- 4)  $\mathbf{DP}_s(K) \subseteq \mathbf{P}_s\mathbf{D}(K)$ ;
- 5)  $\mathbf{DP}_r(K) \subseteq \mathbf{P}_r\mathbf{D}(K)$ ;
- 6)  $\mathbf{DP}_u(K) \subseteq \mathbf{P}_u\mathbf{D}(K)$ ;
- 7)  $\mathbf{DL}(K) \subseteq \mathbf{LD}(K)$ ;
- 8)  $\mathbf{DL}_s(K) \subseteq \mathbf{L}_s\mathbf{D}(K)$ .

*Proof.* Obviously an isomorphism respects the inclusions above.

The inclusions 1) and 2) were proved in [6].

3) and 4) immediately follows from 2).

To show 5) assume that an algebra  $\mathbf{B} \in \mathbf{DP}_r(K)$ . Let  $\mathbf{B} = (A, t_0^\mathbf{A}, t_1^\mathbf{A}, \dots, t_\gamma^\mathbf{A}, \dots)$  is a derived algebra of a reduced product  $\mathbf{A} = (A, f_0^\mathbf{A}, f_1^\mathbf{A}, \dots, f_\gamma^\mathbf{A}, \dots)$ , where  $A = (\Pi_{i \in I} A_i) / \sim_F$ , for a set  $I$ , a family  $(\mathbf{A}_i)_{i \in I}$  of algebras of type  $\tau$  from  $K$  and a filter  $F$  over  $I$ , i.e.  $\mathbf{A} = (\Pi_{i \in I} \mathbf{A}_i) / F$ , where for any  $n$ -ary functional symbol  $f$  the following holds (cf. [5, p. 13]):

$$f^\mathbf{A}(a_0/F, \dots, a_{n-1}/F) = a_n/F \text{ if and only if } \{i \in I : f^{\mathbf{A}_i}(a_0, \dots, a_{n-1}) = a_n\} \in F.$$

Note that by the induction on the complexity of a term  $t$  of type  $\tau$  one may show, that the following holds for any  $n$ -ary polynomial  $t$  of type  $\tau$ :

$$t^\mathbf{A}(a_0/F, \dots, a_{n-1}/F) = a_n/F \text{ if and only if } \{i \in I : t^{\mathbf{A}_i}(a_0, \dots, a_{n-1}) = a_n\} \in F.$$

From the above equality it follows that the algebra  $\mathbf{B} = \Pi_{i \in I} \mathbf{A}_i^\sigma / F$ , where  $\mathbf{A}_i^\sigma = (A_i, t_0^{\mathbf{A}_i}, t_1^{\mathbf{A}_i}, \dots, t_\gamma^{\mathbf{A}_i}, \dots)$ , for  $i \in I$  is a derived algebra of  $\mathbf{A}_i$ , i.e.  $\mathbf{B} \in \mathbf{P}_r\mathbf{D}(K)$ .

The inclusion 6) is an immediate consequence of 5).

To prove 7), let  $\mathbf{B} \in \mathbf{DL}(K)$ , i.e. given a derived algebra  $\mathbf{B}$  of an algebra  $\mathbf{A}$ , where  $\mathbf{A} = (A, f_0^\mathbf{A}, f_1^\mathbf{A}, \dots, f_\gamma^\mathbf{A}, \dots)$  is a direct limit of a triple  $\Lambda = (I, \mathbf{A}_i, g_{ij})$ , where  $I = (I, \leq)$  is an up-directed set,  $(\mathbf{A}_i)_{i \in I}$  is a family of algebras of a given type  $\tau$  and  $\{g_{ij} : i, j \in I, i \leq j\}$  is a family of homomorphisms of  $\mathbf{A}_i$  into  $\mathbf{A}_j$  called a *direct spectrum* over  $(I, \leq)$ , cf. [5, p. 17]. For a given direct spectrum  $\Lambda = (\mathbf{A}_i, g_{ij})$ , we consider the quotient set  $A = \bigcup_{i \in I} A_i \times \{i\} / \equiv$ , where

$$(a, i) \equiv (b, j) \text{ if and only if } (\exists k \in I)(i, j \leq k, g_{ik}(a) = g_{jk}(b)).$$

Let  $\langle a, i \rangle$  denotes the equivalence class by  $\equiv$  containing  $(a, i)$ . The operations on  $A$  are defined by setting for any operation symbol of type  $\tau$ :

$$f^{\mathbf{A}}(\langle a_0, i_0 \rangle, \dots, \langle a_{n-1}, i_{n-1} \rangle) = \langle f^{\mathbf{A}_j}(g_{i_0 j}(a_0), \dots, g_{i_{n-1} j}(a_{n-1})), j \rangle.$$

Note, that as  $g_{ij}$  are homomorphisms for all  $i \leq j$ ,  $i, j \in I$ , therefore for any polynomial symbol  $p$  of type  $\tau$ , the following holds:

$$p^{\mathbf{A}}(\langle a_0, i_0 \rangle, \dots, \langle a_{n-1}, i_{n-1} \rangle) = \langle p^{\mathbf{A}_j}(g_{i_0 j}(a_0), \dots, g_{i_{n-1} j}(a_{n-1})), j \rangle.$$

As  $\mathbf{B} = \mathbf{A}^\sigma$  for a hypersubstitution  $\sigma$  of type  $\tau$ , therefore  $\mathbf{B} \in \mathbf{LD}(K)$ , namely  $\mathbf{B}$  is a direct limit of the triple  $\Lambda^\sigma = (I, \mathbf{A}_i^\sigma, g_{ij})$  of derived algebras of  $K$ , i.e. of a derived spectrum of  $\Lambda$ .

To prove 8), recall [5, 17], that a direct spectrum  $\Lambda = (I, \mathbf{A}_i, g_{ij})$  is called *superdirect* if the mappings  $g_{ij} : A_i \rightarrow A_j$  are surjective. The family  $(\mathbf{A}_i)_{i \in I}$  is referred to as *superdirect family*. The direct limit of a superdirect spectrum is called the *superdirect limit*. Note, that the derived spectrum  $\Lambda^\sigma$  of a superdirect spectrum  $\Lambda$  is superdirect. This together with 7) proves 8).  $\square$

Via corollary 2.3.4 [5, p. 79] and proposition 5.1 we obtain another characterization of hyperquasivarieties:

**Proposition 5.4.** *For any class  $K$  of algebras, the following assertions hold:*

- (1)  $H\mathbf{Q}(K) = \mathbf{SPP}_u\mathbf{D}(K)$ ;
- (2)  $H\mathbf{Q}(K) = \mathbf{SP}_u\mathbf{PD}(K)$ ;
- (3)  $H\mathbf{Q}(K) = \mathbf{SP}_u\mathbf{P}_\omega\mathbf{D}(K)$ ;
- (4)  $H\mathbf{Q}(K) = \mathbf{SL}_s\mathbf{PD}(K) = \mathbf{L}_s\mathbf{SPD}(K) = \mathbf{L}_s\mathbf{P}_s\mathbf{SD}(K)$ .

Via proposition 4.1 we obtain the following reformulation of Mal'cev theorems of [10, p. 214 - 215]:

**Theorem 5.5.** *A class  $K$  of algebras of a given type is a hyperquasivariety if and only if  $K$*

- i) is ultraclosed;*
- ii) is hereditary;*
- iii) is multiplicatively closed;*
- iv) contains a trivial system;*
- v) is derivably closed.*

**Theorem 5.6.** *For every class  $K$  of algebras of a given type we have*

$$H\mathbf{Q}(K) = \mathbf{SP}_r\mathbf{D}(K_0)$$

**Remark.** If we accept, that the direct product of an empty family of algebras of a given type is a trivial algebra of a given type, we may remove condition iv) of theorem 5.4 and substitute  $K_0$  by  $K$  in theorem 5.4. (cf. Corollary 2.3. of [5, p. 78]).

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